Some Remarks on Gravitational Interaction and Gravitational Waves

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A brief consideration of the problem of gravitational waves is given on the basis of the following assumption: The components of the metric tensor are functionals of a field by which, in the sense of HEISENBERG's nonlinear theory, all other fields resp. the corresponding interactions can be deduced. For the sake of mathematical simplicity a scalar field Φ (noncharged bosons) is considered instead of a spinor field. The condition $g_{mn}=g_{mn}(\Phi)$ resp. $R_{mn}=R_{mn}(\Phi)$ leads to the statement that the concept of a free gravitational wave, i. e. a wave which is a solution of $R_{mn}=0$ or $R_{klmn}=0$, cannot be accepted. A free wave is here by definition a wave which is so far from the origin that one can neglect in the field eqs. all terms which represent a strong interaction. A comparison with a spinor field leads, with regard to this definition, to the conclusion that a free wave may be considered as a neutrino wave and gravitation as the weakest interaction possible of neutrino fields.

In two recently published papers [1, 2] concerning a general covariant and nonlinear theory of matter, the following suggestions had been made: (i) The components of the metric tensor are functionals of a so called world field by which, in the sense of HEISENBERG's theory [3], all other fields can be expressed. (ii) To set up a system of field eqs. one can use the eqs. of EINSTEIN choosing the energy tensor in a way that as a consequence of the BIANCHI identities one gets a nonlinear wave equation.

In order to make the here accepted point of view more understandable and to consider the problem of gravitational waves from it, we have first to repeat some results which had been given already in the papers mentioned above. Problems which are connected with the quantization of the field are not touched here. Further, we restrict ourselves to the model of a scalar field (noncharged bosons) – this for the sake of mathematical simplicity. Then we can assume the field eqs. in the following way (comma denotes partial, semicolon covariant differentiation, EINSTEIN's sum convention is used):

$$\frac{1}{\varkappa}\left(R_{mn}-\tfrac{1}{2}g_{mn}R\right) = \tfrac{1}{2}g_{mn}\left(g^{ab}\,\Phi_{,a}\,\Phi_{,b}+F(\Phi)\right)-\Phi_{,m}\,\Phi_{,n}\,. \tag{1}$$

As a consequence of the BIANCHI identities these equations include a nonlinear KLEIN–GORDON equation:

$$g^{ab}\,\Phi_{;ab}-\frac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}\Phi}=0\,. \tag{2}$$

In the static case, which we relate to a single (bare) particle, we have, with a metric which is given by

$$\mathrm{d}s^2=e^{\lambda(r)}\,\mathrm{d}r^2+r^2(\sin^2\Theta\,\mathrm{d}\varphi^2+\mathrm{d}\Theta^2)-e^{\nu(r)}\,c^2\,\mathrm{d}t^2, \tag{3}$$

$$\Phi''+\Phi'\left(\frac{2}{r}+\frac{\nu'-\lambda'}{2}\right)-\frac{1}{2}e^{\lambda}\frac{\mathrm{d}F}{\mathrm{d}\Phi}=0\,,\qquad \Phi'\equiv\frac{\mathrm{d}\Phi}{\mathrm{d}r}\quad\text{etc.}, \tag{4}$$

$$e^{-\lambda}=1-\frac{\varkappa}{2\,r}\,e^{-f}\int_0^r(\Phi'^2+F)\,e^{f}\,r^2\,\mathrm{d}r\,,\qquad f=\frac{\varkappa}{2}\int_0^r\Phi'^2\,r\,\mathrm{d}r\,, \tag{5}$$

$$\nu=-\lambda+\varkappa\int_\infty^r\Phi'^2\,r\,\mathrm{d}r\,; \tag{6}$$

$\varkappa$ is the gravitational coupling constant. One could add in (5) a term with const/r – this term being the exact SCHWARZSCHILD solution of $R_{mn}=0$. In order to fulfill the condition $g_{mn}=g_{mn}(\Phi)$ we have omitted here this term. However, if we choose $F\sim\Phi^6$ we obtain for large values of r for λ and ν a solution which is identical with the SCHWARZSCHILD solution up to terms of higher order. With $F=-(1/3\,l_0^2)\,\Phi^6$, where l_0 denotes a constant which is assumed to be $\approx 10^{-13}$ cm, we find

$$\Phi=A/r+O(1/r^3),\quad A=\text{const},\quad r\gg l_0\,, \tag{7}$$

[1] G. BRAUNSS, Z. Naturforschg. **19 a**, 401 [1964].
[2] G. BRAUNSS, Z. Naturforschg. **19 a**, 1032 [1964].
[3] W. HEISENBERG, Rev. Mod. Phys. **29**, 269 [1957].

$$e^{\lambda}=1+l_G/r+O(1/r^2), \tag{8}$$

$$e^{\nu}=1-l_G/r+O(1/r^2). \tag{9}$$

$l_G=\varkappa\, m_0\, c^2/4+O(\varkappa^2)$ denotes the gravitational radius, $m_0\, c^2$ the energy of the particle.

One can see that the nonlinear term with F may be neglected in the field eqs. as long as one is interested only in interactions at large distances, that is for $r \gg l_0$. The term with F becomes important for small distances therefore apparently representing the strong interactions [4]. On the other hand one can neglect terms with the components of the metric tensor which differ from euclidean values if $r \approx l_0$. So one could set up the following rough scheme:

Small distance $(l_G \ll r \approx l_0)$, strong interaction

Field eqs.:

$$g_{mn} \approx g_{mn}^{\text{Euclid.}},$$

$$0=g^{ab}\, \Phi_{;\,ab}-\frac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}\Phi} \approx \Box\, \Phi-\frac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}\Phi}. \tag{10}$$

Large distance $(r \gg l_0)$, gravitational interaction

Field eqs.:

$$g^{ab}\, \Phi_{,\,a}\, \Phi_{,\,b} \gg F, \tag{11}$$

$$\frac{1}{\varkappa}\left(R_{mn}-\tfrac{1}{2} g_{mn}\, R\right) \approx \tfrac{1}{2} g_{mn}\, g^{ab}\, \Phi_{,\,a}\, \Phi_{,\,b}-\Phi_{,\,m}\, \Phi_{,\,n}.$$

Considering this scheme one can say that gravitational interaction is the weakest interaction possible. Now, going back to the eqs. (8) and (9) we may say that we have obtained the wellknown result of the theory of general relativity with respect to the SCHWARZSCHILD metric despite the fact that we did not use the equations of an empty space but these of an nonempty space (namely filled with a field Φ). This is in another way expressed by the circumstance that we do not consider the g_{mn} as components of a field but rather as functionals of a field. The conception of an empty space, i. e. $R_{mn}=0$, has no sense if one accepts the field character of matter. For the same reasons we cannot accept the conception of free gravitational waves if one means by that waves which are solutions of $R_{mn}=0$. Gravitation means a certain type of interaction and not a field itself. What we may look for is the interaction of Φ-waves. In a certain sense we may speak of a free wave if we consider such waves at large distances from their "centers", that is, if we neglect the terms which represent the strong interactions. In this case we get from (11)

$$g^{ab}\, \Phi_{;\,ab}=0. \tag{12}$$

Let us consider for a moment a spinor field instead of a scalar field. Then (12) would correspond to a neutrino equation which means that a free wave is a neutrino wave and gravitation the (weakest) interaction of neutrinos [5].

We try now to get a first approximation of these so defined free waves. Since we are looking for spherical waves we can assume the following metric:

$$\mathrm{d}s^2=e^{\lambda(r,t)}\, \mathrm{d}r^2+r^2\, (\sin^2\Theta\, \mathrm{d}\varphi^2+\mathrm{d}\Theta^2) - e^{\nu(r,t)}\, c^2\, \mathrm{d}t^2. \tag{13}$$

For the field equations we find with the abbreviations $\Phi' \equiv \dfrac{\partial \Phi}{\partial r}$, $\dot{\Phi} \equiv \dfrac{\partial \Phi}{c\, \partial t}$ etc.:

$$e^{-\lambda}\left[\Phi''+\Phi'\left(\frac{2}{r}+\frac{\nu'-\lambda'}{2}\right)\right] - e^{-\nu}\left(\ddot{\Phi}+\dot{\Phi}\,\frac{\dot{\lambda}-\dot{\nu}}{2}\right)-\frac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}\Phi}=0, \tag{14}$$

$$(e^{-\lambda})'+e^{-\lambda}\left(\frac{1}{r}+\frac{1}{2}\varkappa\, r\, \Phi'^2\right) -\frac{1}{r}+\frac{1}{2}\varkappa\, r\,(e^{-\nu}\, \dot{\Phi}^2+F)=0, \tag{15}$$

$$(e^{-\lambda})'+e^{-\lambda}(-\nu'+\varkappa\, r\, \Phi'^2)+\varkappa\, r\, e^{-\nu}\, \dot{\Phi}^2=0. \tag{16}$$

Leaving out particular solutions of $R_{mn}=0$ we can write the first two eqs. in the following way (the constants of integration are taken in the same way as in the static case):

$$e^{-\lambda}=1-\frac{\varkappa}{2\,r}\int_0^r (e^{-\lambda}\, \Phi'^2+e^{-\nu}\, \dot{\Phi}^2+F)\, r^2\, \mathrm{d}r, \tag{17}$$

$$\nu=-\lambda+\varkappa\int_\infty^r (\Phi'^2+e^{\lambda-\nu}\, \dot{\Phi}^2)\, r\, \mathrm{d}r. \tag{18}$$

We note that

$$-T_4{}^4=\tfrac{1}{2}\,(e^{-\lambda}\, \Phi'^2+e^{-\nu}\, \dot{\Phi}^2+F) \tag{19}$$

is the energy density.

[4] The classification "weak" and "strong" is not a very happy one because at large distances the strong interactions become weak while the very weak (gravitational) interaction becomes strong. On the other hand, concerning the behaviour of Φ on the light cone, there are some reasons to believe that the terms which represent the gravitational interaction cannot be neglected within regions which are of the order of $l_G \ll l_0$. A reasonable classification of interactions should be based upon an investigation of the different types of nonlinearities.

[5] The idea of representing a gravitational field by neutrino fields has been suggested i. a. by P. M. A. DIRAC, see Max-Planck-Festschrift 1958, p. 339.

Now, since we are interested in weak interactions at large distances, we can neglect all terms with F. Further, we may neglect approximately all terms which are of the order of $\varkappa^2$. Then we get

$$\lambda = \frac{\varkappa}{2r} \int\limits_0^r (\Phi'^2 + \dot{\Phi}^2)\, r^2\, \mathrm{d}r + O(\varkappa^2), \tag{20}$$

$$\nu = -\lambda + \varkappa \int\limits_\infty^r (\Phi'^2 + \dot{\Phi}^2)\, r\, \mathrm{d}r + O(\varkappa^2). \tag{21}$$

Substituting

$$\tfrac{1}{2}(\lambda - \nu) = \alpha, \quad \tfrac{1}{2}(\lambda + \nu) = \beta, \tag{22}$$

we obtain finally from (14), again neglecting terms with $\varkappa^2$,

$$\Phi'' + \frac{2}{r}\Phi' - e^{2\alpha}\ddot{\Phi} - \dot{\alpha}\dot{\Phi} - \alpha'\Phi' = 0, \tag{23}$$

$$(\alpha r)'' = -\beta' = -\tfrac{1}{2}\varkappa r(\Phi'^2 + \dot{\Phi}^2). \tag{24}$$

To get a first approximation we neglect in (23) all terms with α. If we permit only such solutions which correspond to outgoing waves we have the well-known expression for spherical waves:

$$\Phi_{(0)} = \frac{\vartheta_{(0)}(r - c\,t)}{r}. \tag{25}$$

Let us consider the case of a limited wave with the following initial conditions

$$\vartheta'^2_{(0)} = \dot{\vartheta}^2_{(0)} = \begin{cases} \vartheta_{\mathrm{M}}^2 + P(r - c\,t) & \text{if} \quad r \leqq R = c(t + t_0), \\ 0 & \text{if} \quad r > R; \end{cases} \tag{26}$$

$\vartheta_{\mathrm{M}} = \text{const.}$ denotes a mean value and $P(r - c\,t)$ a periodical function for which $-\vartheta_{\mathrm{M}}^2 \leqq P(r - c\,t) \leqq \vartheta_{\mathrm{M}}^2$ holds. For the sake of simplicity we assume $\vartheta_{\mathrm{M}}^2 = 1$. Then we obtain with (24) and (25)

$$\alpha = \tfrac{1}{2}(\lambda - \nu) = -\varkappa \log r + C_1 + O(1/r), \tag{27}$$

$$\beta = \tfrac{1}{2}(\lambda + \nu) = \varkappa \log r + C_2 + O(1/r). \tag{28}$$

Since no interaction can have a velocity greater than c, we must require the vanishing of α at the wave front which gives

$$\alpha(R) = -\varkappa \log R + C_1 + O(1/R) \tag{29}$$

resp. with $R \gg 1$

$$C_1 \approx \varkappa \log R \tag{30}$$

and thus

$$\alpha = -\varkappa \log(r/R) + O(1/r). \tag{31}$$

Substituting this expression for α in eq. (23) we get the equation

$$\Phi'' + \frac{2}{r}\Phi' - \left(\frac{r}{R}\right)^{-2\varkappa}\ddot{\Phi} - \frac{\varkappa}{R}\dot{\Phi} + \frac{\varkappa}{r}\Phi' = 0, \tag{32}$$

which we can use to obtain a better approximation for Φ (if we take the convergence for granted). Using the method of separation we get with

$$\Phi = \frac{1}{r} h(r)\, g(R), \quad r \leqq R = c(t + t_0), \tag{33}$$

the following eqs. (neglecting again terms with $\varkappa^2$):

$$h'' + \frac{\varkappa}{r} h' + \left(\frac{k^2}{r^{2\varkappa}} - \frac{\varkappa}{r^2}\right) h = 0, \quad k = \text{const}, \tag{34}$$

$$\ddot{g} + \frac{\varkappa}{R}\dot{g} + \frac{k^2}{R^{2\varkappa}} g = 0. \tag{35}$$

In the first equation the second term in the brackets is small compared to the first term so we can neglect it. Then we get the following solutions expressed by BESSEL functions:

$$h(r) = r^q Z_p(\mu r^\gamma), \quad 2q = \gamma = 1 - \varkappa, \quad \mu\gamma = k, \tag{36}$$

$$g(R) = R^q Z_p(\mu R^\gamma); \tag{37}$$

k is a constant of integration. If we choose $k^2 > 0$ we obtain periodical solutions. Going back to (33) and using the representation of BESSEL functions in the case of periodical solutions we finally get up to terms of higher order

$$\Phi_{(1)} = \frac{1}{r^{1+\varkappa}}\left(\frac{r}{R}\right)^{\varkappa/2} \vartheta_{(1)}(r^{1-\varkappa} - R^{1-\varkappa}); \tag{38}$$

$\vartheta_{(1)}(r^{1-\varkappa} - R^{1-\varkappa})$ denotes a periodical function of its argument which shall vanish identically if $r > R$.

This result shows that we have a dispersion with a velocity given approximately by

$$r^{1-\varkappa} - R^{1-\varkappa} = \text{const}, \qquad R = c(t + t_0), \tag{39}$$

resp.

$$\frac{v}{c} = \frac{\mathrm{d}r}{c\,\mathrm{d}t} = \left(\frac{r}{R}\right)^{\varkappa}, \quad r \leqq R. \tag{40}$$

The same result could have been obtained immediately with the help of (29) from the equation of null geodesics. It shows that the wave front $r = R$ moves with the velocity of light while inside the region $r < R$ the velocity falls off according to (40). Hence a wave train which started with a certain wave length L has after a time Δt in the example chosen here a length $L + \Delta L$ with $\Delta L \approx 2\varkappa c\,\Delta t$. This corresponds to a loss of energy per length which in our example is approximately given by $\Delta E \approx 1 - (r/R)^{2\varkappa}$. In the usual picture the foregoing procedure may be considered as the calculation of a graviton-graviton interaction.

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